NEW APPROACH TO KPZ MODELS THROUGH FREE FERMIONS AT POSITIVE TEMPERATURE

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Abstract. We give a short account of our new approach to study models in the Kardar-Parisi-Zhang(KPZ) universality class by connecting them to free fermions at positive temperature. Our ideas and methods are explained mainly for the semi-discrete directed polymer model due to O'Connell and Yor.

Contents

1. Introduction

Non-equilibrium statistical physics has been an important field in mathematical physics. For systems in thermal equilibrium, there is a standard formulation using Gibbs measure for given Hamiltonians [Rue69]. Systems far from equilibrium show rich intriguing phenomena, which are not seen in equilibrium systems, such as dissipative structures. Many non-equilibrium systems are formulated as stochastic interacting systems, which are stochastic processes with infinite degrees of freedom with interaction. Foundations of such systems have been established in 1970's and 80's [Spi70, Lig85]. There have been accumulation of results since then [Lig99, KL99]. Recently particular class of interacting particle systems attract special attention, which are exactly solvable. They have connections with integrable systems [Bax82], which allow to study various properties in detail. The subject is often called integrable probability [BP16].

One of the central objects in this field is the models in the Kardar-Parisi-Zhang(KPZ) universality class [KPZ86, BS95], which share various non-trivial fluctuation properties. We call them the KPZ models in this article. The name of the universality class originates from the equation with the same name, which was introduced in 1986 by

Kardar, Parisi and Zhang to describe interface motion [KPZ86]. There are interests for models in general dimension, but there have been a big progress for one dimensional case because some KPZ models in one dimension can be studied explicitly. The KPZ equation in one dimension is the following. For the height function $h = h(x,t), x \in \mathbb{R}, t \in \mathbb{R}_+$,

$$\frac{\partial}{\partial t} h = \frac{1}{2}\frac{\partial^2}{\partial x^2} h + \frac{1}{2}\left(\frac{\partial h}{\partial x}\right)^2 + \eta \tag{1.1}$$

where $\eta = \eta(x,t)$ is the space time white noise. On the right hand side, the first term represents a smoothening, the second a nonlinearity coming from vertical growth and the last a noise from the environment. See [BS95, Cor12, Sas16].

It is often useful to apply the Cole-Hopf transformation: $Z = Z(x,t) = e^{h(x,t)}$. The KPZ equation becomes

$$\frac{\partial}{\partial t} Z = \frac{1}{2}\frac{\partial^2}{\partial x^2} Z + \eta Z. \tag{1.2}$$

From this equation, Z may be interpreted as the partition function for a directed polymer in random environment η at a finite temperature ($T > 0$). As written there is an issue of well-definedness for (1.1). It was shown in [BG97] that if one interprets the product ηZ in (1.2) as Ito type, one can make sense of (1.1). Now there are a few ways to define (1.1) and make sense of more general nonlinear stochastic PDEs [Hai13, GP17, Kup16], but we don't go into details of this aspect in this article.

As mentioned above, a striking feature of the one dimensional KPZ equation is that it admits exact analysis. In 2010 the explicit formula for the point distribution of the height at one point was discovered for the narrow wedge initial condition [SS10d, SS10a, SS10b, SS10c, ACQ11]. In terms of Z, result reads the following.

Theorem 1.1. *For the initial condition $Z(x,0) = \delta(x)$, the Laplace transform of $Z(0,t)$ is written as the Fredholm determinant,*

$$\mathbb{E}[\exp(-Z(0,t)e^{\frac{t}{24}-(t/2)^{1/3}s})] = \det(1-K_t)_{L^2(\mathbb{R}_+)}, \tag{1.3}$$

where the kernel is given by

$$K_t(x,y) = \int_{\mathbb{R}} \frac{\mathrm{Ai}(x+\lambda)\mathrm{Ai}(y+\lambda)}{1+e^{(t/2)^{1/3}(s-\lambda)}} d\lambda. \tag{1.4}$$

Here Ai *is the Airy function.*

Using this formula it is rather straightforward to do asymptoics and establish that the limiting distribution of the free energy $\log Z(0,t)$ obeys the Tracy-Widom distribution [TW94] in the large t limit.

In the kernel, we see the Fermi-Dirac factor of the form, $1/(1+e^{\beta(\epsilon-\mu)})$. This suggests that this is related to a free fermion at finite temperature. See Appendix A. We emphasize that the formula (1.3) with (1.4) had not been obtained through a direct relation between the KPZ equation and a free fermion. In [SS10a, ACQ11], the formula was first found by taking a limit for the results for the asymmetric simple exclusion process(ASEP) by Tracy and Widom [TW09], which had been obtained by using Bethe ansatz. Soon afterwards the same formula was reproduced by using replica calculations, where the Lieb-Liniger model appears and is solved using again the Bethe ansatz [CDR10, Dot10]. But direct connection between the KPZ models and free fermion at finite temperature had not been established. A natural question "Can we find a connection between the KPZ equation and a free fermion at finite temperature before getting formulas by lengthy Bethe ansatz calculations?" had remained for more than a decade.

In [IMSa, IMSb], we found such a connection for discretized KPZ models. More precisely we found a bijection which connects the q-Whittaker measure and the periodic Schur measure. The q-Whittaker measure has been known to be related to KPZ models [BC14, BP16]. The periodic Schur measure is associated with a free fermion at positive temperature [Bor07, BB19]. Then, using this connection, one can study KPZ models associated with the q-Whittaker measure with standard machinery of free fermions. By considering appropriate special and limiting cases, one can study various KPZ models with free fermion techniques, including the KPZ equation itself. Hence we now have a route to arrive at the formula (1.3) without going through Bethe ansatz calculations. Applications of our new approach to KPZ models are discussed in [IMSc]. We remark that a relation between discretized KPZ models and the Schur measure had been discussed by Borodin [Bor18], but it was through a matching of expectations and was not directly related to a finite temperature free fermion.

In this article we explain our approach to KPZ models through free fermions at finite temperature. We mainly focus on a particular example of finite temperature directed polymer model known as the O'Connell-Yor polymer model [OY01].

The rest of this article is organized as follows. In the next section, we introduce the O'Connell-Yor polymer model and state the Fredholm determinant formula for its Laplace transform which appear from our approach. In section 3, the simpler case of zero temperature is explained. In section 4, the relation to quantum Toda lattice and Whittaker measure are explained. In section 5, we give a brief account of our results on the q-PushTASEP, q-Whittaker measure and periodic Schur measure. A Fredholm determinant formula for q-PushTASEP is presented. In section 6 we take appropriate $q \to 1$ limit of both sides of the identity to arrive at the formula presented in section 2. Conclusion is given in section 7. In Appendix A basics of determinantal point process(DPP) and free fermions are given. In Appendix B, discrete directed polymer at zero temperature and its relation to TASEP are explained.

2. O'Connell-Yor polymer

2.1. **Model.** The O'Connell-Yor polymer model is a directed polymer model defined on semi-discrete setting (discrete space and continuous time), which was introduced by O'Connell and Yor [OY01]. Consider N semi-infinite lines $\{(j,s), s \geq 0\}, j = 1, 2, \dots, N$ and a path π which consists of segments $(j, t_{j-1}) \to (j, t_j)$, $j = 1, 2, \dots, N-1$, $0 = t_0 < t_1 < \ldots < t_{N-1} < t_N = t$. One can regard π as a polymer which starts from $(0,0)$, switches from the jth line to the $(j+1)$th line at time t_j for $j = 1, \dots, N-1$, and ends at (N,t), see Fig 1. Suppose each line is associated with a Brownian motion, B_j, which are independent, and the polymer feels the potential energy from them on each

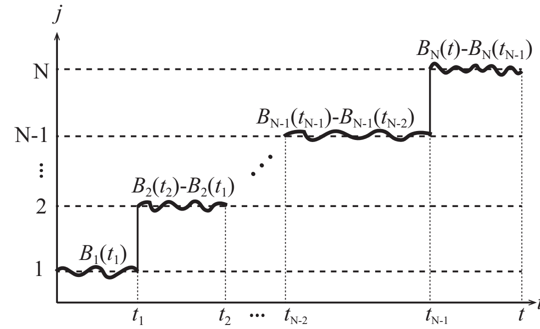

FIGURE 1. The O'Connell-Yor polymer.

segment, s.t., the energy of the polymer is given by

$$E[\pi] = \sum_{i=1}^{N} (B_i(t_i) - B_i(t_{i-1})). \tag{2.1}$$

The partition function is given by

$$Z_N(t) = \int_{0<t_1<\ldots<t_{N-1}<t} e^{\beta E[\pi]} dt_1 \cdots dt_{N-1}. \tag{2.2}$$

Here $\beta = 1/k_B T$ is the inverse temperature with k_B the Boltzmann constant and $T(>0)$ the temperature. In this article we set $\beta = 1$ except at the beginning of section 3 where we discuss the zero temperature limit. $Z_N(t)$ is random because $E[\pi]$ is.

One can introduce $Z_j(t), 1 \le j \le N$ by considering the first j lines in the above. By Ito's formula they satisfy

$$dZ_j(t) = Z_{j-1}(t)dt + Z_j(t)dB_j(t), \tag{2.3}$$

where $Z_0(t) = 0$ by convention and we interpret the product in the second term as Ito type. In an appropriate continuous space limit, this becomes the directed polymer for the KPZ equation (1.2).

2.2. **Fredholm determinant formula for Laplace transform.** The goal of this article is to explain how our approach leads to the following Fredholm determinant formula for the Laplace transform of $Z_N(t)$.

Theorem 2.1. *For $s \in \mathbb{R}$ we have*

$$\mathbb{E}[\exp(e^{-s} Z_N(t))] = \det(1-K)_{L^2(s,\infty)}, \tag{2.4}$$

where

$$K(x,y) = \int_{\mathrm{i}\mathbb{R}-d} \frac{\mathrm{d}z}{2\pi\mathrm{i}} \int_{\mathrm{i}\mathbb{R}+d'} \frac{\mathrm{d}w}{2\pi\mathrm{i}} \frac{\pi}{\sin(\pi(w-z))} \frac{\Gamma(z)^N}{\Gamma(w)^N} e^{w^2t/2 - z^2t/2 + zx - wy}, \tag{2.5}$$

with $d, d' > 0$, such that $\frac{1}{2N} < d' + d < 1$.

In a certain limit, (2.4) and (2.5) tend to (1.3), (1.4) for the KPZ equation. With this formula, performing asymptotics is rather easy and one can establish the Tracy-Widom distribution for the O'Connell-Yor polymer [BC14, BCR13]. See the discussion for the case of Log-Gamma polymer in [IMSc]. The formula of Theorem 2.1 seems new, though a few related ones have been written for instance in [BC14, BCR13, IS17].

3. Zero temperature case

Before going to more details about the finite temperature polymer, we discuss the simpler case of zero temperature in this section. They would be also useful to understand the finite temperature case.

In the zero temperature ($\beta \to \infty$) limit, the free energy becomes the ground state energy,

$$\lim_{\beta\to\infty} \beta \log Z_N(t) = \sup_\pi E[\pi]. \tag{3.1}$$

In [Bar01, GTW02], it was shown that this is related to Gaussian unitary ensemble (GUE) from random matrix theory [Meh04, For10]. More precisely, $\sup_\pi E[\pi]$ has the same law as the largest eigenvalue x_1 of GUE of size N, i.e.,

$$\mathbb{P}[\sup_\pi E[\pi] \le s] = \mathbb{P}[x_1 \le s]. \tag{3.2}$$

The probability density function of N eigenvalues of GUE is written as

$$\frac{1}{Z} \prod_{1\le j<k\le N} (x_j - x_k)^2 \prod_{j=1}^N e^{-x_j^2}. \tag{3.3}$$

Here and in the following Z denotes a normalization constant and may differ from one case to another. Using the multi-linearity of determinants, this can be rewritten as $\frac{1}{Z}\det(\phi_j(x_k))^2$ where $\phi_n(x) = \frac{e^{-x^2/2}}{\pi^{1/4}\sqrt{2^n n!}} H_n(x)$ and $H_n(x)$ is the Hermite polynomial [AAR99]. This can be understood as the probability density of N particles of free fermions under harmonic potential in the ground state. See Appendix A. By the standard methods of DPP, the probability $\mathbb{P}[x_1 \le s]$ is written as the Fredholm determinant,

$$\mathbb{P}[x_1 \le s] = \det(1-K)_{L^2(s,\infty)}, \tag{3.4}$$

where

$$K(x,y) = \sum_{n=0}^{N-1} \phi_n(x)\phi_n(y). \tag{3.5}$$

Combining (3.2) and (3.4), we find

$$\mathbb{P}[\sup_\pi E[\pi] \le s] = \det(1-K)_{L^2(s,\infty)} \tag{3.6}$$

with K given by (3.5). This is nothing but the zero temperature limit of the relation (2.4). Indeed the left hand side of (2.4) becomes the distribution function of the ground state energy. In the right hand side $\Gamma(z), \Gamma(w)$ are replaced by z, w and $\sin\pi(w-z)$ by $\pi(w-z)$. The result is a well-known double contour integral formula of the kernel (3.5).

One way to understand this relation between the polymer and the GUE is to consider a Markov dynamics on a Gelfand-Tsetlin(GT) cone. See Fig. 2. In the continuous setting, this may be considered as positions of particles $x_i^{(k)} \in \mathbb{R}, 1 \le i \le k \le N$,

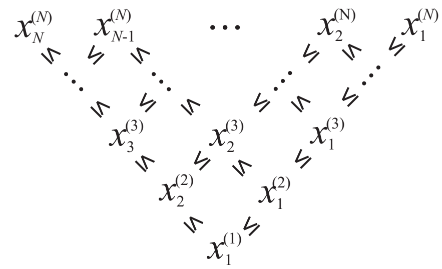

FIGURE 2. The Gelfand-Tsetlin cone as a triangular array.

satisfying the relations $x_{i+1}^{(k+1)} \leq x_i^{(k)} \leq x_i^{(k+1)}$. We may consider a Markov dynamics of these particles. First $x_1^{(1)}$ is a Brownian motion. Other particles are also Brownian motions but they should satisfy the condition of the GT cone. If one focuses on $x_k^{(k)}, 1 \leq k \leq N$ on the diagonal on the left edge of the GT cone, this is the minimum energy $\min_\pi E[\pi]$ of the polymer up to the k-th level. On the top we have Dyson's Brownian motion, as shown in [War07], see also [Sas11, WW09]. At a fixed time they are the GUE eigenvalues.

A similar discussion for discrete directed polymer models and its connection to TASEP will be explained in Appendix B.

4. Relation to Whittaker measure

Now we come back to the discussion about the O'Conne-Yor polymer.

4.1. Gelfand Tsetlin dynamics for O'Connel-Yor polymer. In [O'C12], O'Connell found a generalization of the Markov dynamics on the GT cone at zero temperature in the previous section to the case of finite temperature. With $B_k, k = 1, \cdots, N$ independent Brownian motions, one can now consider $F_i^{(k)}, 1 \leq i \leq k \leq N$ satisfying $dF_1^{(1)} = dB_1$ and

$$
\begin{aligned}
dF_1^{(k)} &= dF_1^{(k-1)} + e^{F_2^{(k)} - F_1^{(k-1)}} dt, \\
dF_2^{(k)} &= dF_2^{(k-1)} + (e^{F_3^{(k)} - F_2^{(k-1)}} - e^{F_2^{(k)} - F_1^{(k-1)}}) dt, \\
&\cdots \\
dF_{k-1}^{(k)} &= dF_{k-1}^{(k-1)} + (e^{F_k^{(k)} - F_{k-1}^{(k-1)}} - e^{F_{k-1}^{(k)} - F_1^{(k-2)}}) dt, \\
dF_k^{(k)} &= -dB_k - e^{F_k^{(k)} - F_{k-1}^{(k-1)}} dt, \qquad (4.1)
\end{aligned}
$$

for $k = 2, \ldots, N$. One observes that $F_k^{(k)}$ evolves autonomously. Moreover, if we set $Z_k := e^{-F_k^{(k)}}$, they satisfy (2.3). Namely, the diagonal elements on the left edge correspond to the O'Connell-Yor polymer. Notice that $F_i^{(k)}$ do not have to satisfy the interlacing conditions. It is satisfied only in the the zero temperature limit, in which the dynamics reduces to the one in the previous section [O'C12].

4.2. **Quantum Toda lattice and Whittaker measure.** In section 3, we saw that, in the zero temperature case, the top particles evolve as the Dyson's Brownian motion. Remarkably in [O'C12], it was shown that the top particles $\{F_k^{(N)}, 1 \le k \le N\}$ of (4.1) also evolve autonomously with some generator denoted by L. It was further shown that, by a similarity transformation, L is mapped to a self-adjoint operator H, given by

$$H = -\sum_{i=1}^{N} \frac{\partial^2}{\partial x_i^2} + \sum_{i=1}^{N-1} e^{x_{i+1}-x_i}, \tag{4.2}$$

which is nothing less than the Hamiltonian of the quantum Toda lattice. For the classical case, the Toda lattice is one of the most well known and important integrable systems. It has soliton solutions and has a number of applications to several real physical systems [Tod89]. The quantum version has been less studied but is known to possess rich and nice properties. As is clear from the Hamiltonian in (4.2), there are interaction between neighboring particles through exponential potential. The system is clearly not free. But the quantum Toda lattice is known as a quantum integrable system. It can be solved by Bethe ansatz and there is a set of operators including the Hamiltonian which mutually commute.

The eigenfunctions of the quantum Toda lattice are known. They are called the Whittaker function $\Psi_\lambda(x)$ with $x \in \mathbb{R}^N, \lambda \in \mathbb{R}^N$ and satisfy

$$H\Psi_\lambda(x) = \left(\sum_{j=1}^{N} \lambda_j^2\right) \Psi_\lambda(x). \tag{4.3}$$

The probability density of the particles on the top line in the GT dynamics above can be written in terms of the Whittaker function. This measure on $\mathbb{R}^N$ is called the Whittaker measure and is written in the form,

$$\frac{1}{Z}\Psi_0(x)\theta_t(x). \tag{4.4}$$

The other function θ_t is a "dual" Whittaker function given by

$$\theta_t(x) = \int_{(i\mathbb{R})^N} d\lambda \cdot \Psi_{-\lambda}(x) e^{\sum_{j=1}^N \lambda_j^2 t/2} s_N(\lambda), \tag{4.5}$$

where $s_N(\lambda)d\lambda$ is the Sklyanin measure,

$$s_N(\lambda) = \frac{1}{(2\pi i)^N N!} \prod_{i<j} \frac{\sin \pi(\lambda_i - \lambda_j)}{\pi} \prod_{i>j} (\lambda_i - \lambda_j). \tag{4.6}$$

See remark at the end of 5.1. In the zero temperature limit, the measure tends to the GUE measure (3.3), see Sec.6 in [O'C12].

4.3. **O'Connell-Yor polymer and Whittaker meaure.** Using the fact that $Z_N(t)$ is also x_1 in the Whittaker measure, one can write down a formula [O'C12],

$$\mathbb{E}[\exp(-e^{-s} Z_N(t))] = \frac{1}{Z} \int_{\mathbb{R}^N} e^{-e^{x_1 - s}} \Psi_0(x)\theta_t(x) dx. \tag{4.7}$$

Corresponding to the fact that the quantum Toda lattice is not a free system, there is no known formula as a single determinant for the Whittaker function. Contrary to the zero temperature case, one can not apply the machinery of DPP. But using the integrability of the Toda lattice, one can do various calculations and arrive at a Fredholm determinant formula. See for example [BC14, BCR13, IS16].

5. q-Whittaker measure and periodic Schur measure

After the formula (1.3) was discovered for the KPZ equation, various generalizations were achieved. The most successful direction was to find and solve discrete KPZ models. Many new models were invented and solved such as the q-TASEP, ASEP, q-Hahn TASEP[BC14, BCS14]. The notable one was the stochastic higher spin six vertex model [CP16], which turns out to contain most of the known models as special and limiting cases. What has turned out was that most models are related to the q-Whittaker measure. (Some models are more directly related to the Hall-Littlewood measure, but here we do not make a clear distinction between them because the latter may be realized as a particular specialization of the former. See discussions in section 2.4 of [IMSc].) The connection between the KPZ models and the q-Whittaker measure is seen through the branching rule of q-Whittaker function, in a similar way as the O'Connell-Yor polymer is related to the Whittaker measure as explained in the previous section. The q-Whittaker function is again not written as a single determinant and is not directly related to a free fermion system. But using the Bethe ansatz or Macdonald operator, one can study the q-Whittaker measure exactly, and one obtains a Fredholm determinant for discrete KPZ models [BC14, BCS14]. But asymptotics is somewhat involved and half-space case has been difficult to analyze.

In [IMSa, IMSb], we have succeeded in finding a bijective connection between the q-Whittaker measure and the periodic Schur measure. The latter is related to a free fermion at positive temperature. Then one can study the KPZ models related to the q-Whittaker measure by the standard techniques for free fermion systems. In this section we briefly review this connection.

5.1. **q-TASEP and q-Whittaker measure.** The most relevant KPZ model in the discussion is the q-PushTASEP, introduced in [MP17]. This is a generalization of the discrete time TASEP, see Appendix B, and is also in the KPZ class. The dynamical rules of this model is rather involved and we do not explain them here. The most important property of the q-PushTASEP in our discussion is that the position of the N-th particle has the same distribution as N plus μ_1 in the q-Whittaker measure of the form,

$$\frac{1}{Z}P_\mu(a)Q_\mu(b), \tag{5.1}$$

where $P_\mu(a)$ and $Q_\mu(a)$ are q-Whittaker functions and they are related by $Q_\mu(a) = b_\mu(q)P_\mu(a)$ with $b_\mu(q) = 1/\prod_{i\geq 1}(q;q)_{\mu_i-\mu_{i+1}}$. In an appropriate $q \to 1$ limit, P_μ tends to the Whittaker function. In this limit the simple relation between P_μ and Q_μ is lost but one can check Q_μ tends to θ_t in (4.5). See Remark 4.1.17 in [BC14].

5.2. **Periodic Schur measure.** Periodic Schur measure was introduced by Borodin in 2007 in [Bor07] and is written in the form,

$$\frac{1}{Z}\sum_{\rho\in\mathcal{P},\rho(\subset\lambda)} q^{|\rho|}s_{\lambda/\rho}(a)s_{\lambda/\rho}(b), \tag{5.2}$$

where $s_{\lambda/\rho}$ is the skew Schur function. This is not a DPP but its shift mixed version $(\lambda_i \to \lambda_i + S)$ with

$$\mathbb{P}(S=\ell) = \frac{t^\ell q^{\ell^2/2}}{(q;q)_\infty \theta(-tq^{1/2})}, \qquad \ell\in\mathbb{Z},\ \text{for } t>0 \tag{5.3}$$

with $\theta(x) = (x;q)_\infty (q/x;q)_\infty$, is a DPP. Later Betea and Bouttier reformulated the periodic Schur measure in terms of free fermion at finite temperature [BB19].

5.3. **Relation of the two measures.** While the q-Whittaker measure in 5.1 is not a DPP, the periodic Schur measure in 5.2 is. As such apparently there seem no relation between the two measures. However, in [IMSa, IMSb], we found that there is a clear and solid relation between the two measures.

Theorem 5.1. *Let $\mathbb{E}$ denote the expectation with respect to the q-Whittaker measure (5.1) and $\mathbb{P}$ the probability measure with respect to the periodic Schur measure (5.2). The following equivalence holds:*

$$\mathbb{E}\left[1/(-tq^{\frac{1}{2}+n-\mu_1};q)_\infty\right] = \mathbb{P}(\lambda_1 + S \le n). \tag{5.4}$$

This means that one can study the distribution of a particle in q-PushTASEP by free fermions at positive temperature.

The theorem was first proved by matching the Fredholm determinants of both hand sides in [IMSa]. But in [IMSb] we gave a completely different combinatorial proof without using Bethe ansatz type calculations. In terms of the q-Whittaker and skew Schur function, the relation (5.4) is equivalent to the following identity,

$$\sum_{\ell=0}^{n} \frac{q^\ell}{(q;q)_\ell} \sum_{\mu:\mu_1=n-\ell} b_\mu(q)P_\mu(a)P_\mu(b) = \sum_{\lambda,\rho:\lambda_1=n} q^{|\rho|}s_{\lambda/\rho}(a)s_{\lambda/\rho}(b). \tag{5.5}$$

5.4. **Bijective proof in** [IMSb]**.** In Appendix B, it was explained that the TASEP is related to the Schur measure by using RSK correspondence and that the Schur function appears when one takes a sum over semi-standard Young tableaux with a given shape. A similar idea may be applied here to prove (5.5) bijectively.

A combinatorial formula for the skew Schur function on the right hand side is well known and reads

$$s_{\lambda/\rho}(x) = \sum_{T\in \mathrm{SST}(\lambda/\rho)} x^T, \tag{5.6}$$

where $x^T = \prod_i x_i^{\#i \text{ in } T}$ and $\mathrm{SST}(\lambda/\rho)$ is the set of skew semistandard tableaux with skew shape λ/ρ. For the q-Whittaker function on the left hand side, a definition using branching rule is commonly used in the field but there is a combinatorial formula due to [San00], which turns out to be useful in our discussion. Its reads

$$P_\mu(x) = \sum_{V \in \mathrm{VST}(\mu)} q^{H(V)} x^V, \tag{5.7}$$

where VST(μ) is the set of "vertically strict tableaux" with increasing elements in each column and no condition among columns. $H(V)$ is the energy function depending on a VST V [NY97]. We do not give a precise definition here but roughly it measures how far the VST V is from a semi-standard tableaux.

Theorem 5.2. *There is a bijection $\Upsilon : (P, Q) \leftrightarrow (V, W, \kappa, \nu)$ where (P, Q) is a pair of skew SSTs with same shape λ/ρ, (V, W) is a pair of VSTs with same shape μ, ν is a partition and*

$$\kappa \in \mathcal{K}(\mu) = \{\kappa = (\kappa_1, \dots, \kappa_{\mu_1}) \in \mathbb{N}_0^{\mu_1} : \kappa_i \geq \kappa_{i+1} \text{ if } \mu'_i = \mu'_{i+1}\}.$$

The bijection Υ has the weight preserving property,

$$|\rho| = H(V) + H(W) + |\kappa| + |\nu|. \tag{5.8}$$

Once we establish this theorem, showing (5.5) is rather simple by noting $\sum_{\kappa \in \mathcal{K}(\mu)} q^{|\kappa|} = b_\mu(q)$ and $\mathbb{P}[\nu_1 = \ell] = \frac{q^\ell}{(q;q)_\ell}(q; q)_\infty$.

In [IMSb], the bijection was constructed by introducing a deterministic time evolution as iterations of skew RSK maps introduced by Sagan and Stanley in [SS90]. We call this the skew RSK dynamics. According to this dynamics, a pair of skew tableaux (P, Q) evolves into a different pair of the same shape. After long time, it shows particular property that all columns proceed with fixed speeds, which resemble solitons in Box and Ball systems [TS90, IKT12].

To give an actual proof of the bijection based on the rules of RSK correspondence seems difficult. In [IMSb] we employed the theory of crystal [Kas90, Kas91, Lus90] to study systematically properties of skew RSK dynamics. We found that a novel realization of affine crystal commutes with the skew RKS dynamics and allows us to study it through a particularly simple tableaux for which the dynamics is linearized. In this short article we do not go into more details. An interested reader is invited to read [IMSb], in particular its introduction at first.

5.5. **Fredholm determinat formula for q-PushTASEP.** As already mentioned, the periodic Schur measure is associated with a free fermion at finite temperature and its shift mixed version is a DPP. By standard methods, one can write down a Fredholm determinant formula for the distribution of λ_1. It reads

$$\mathbb{P}[\lambda_1 + S \leq s] = \det(1 - K)_{\ell^2(\mathbb{Z}_{>s})}, \tag{5.9}$$

where

$$K(x, y) = \frac{1}{(2\pi \mathrm{i})^2} \oint_{|z|=r} \frac{\mathrm{d}z}{z^{x+1}} \oint_{|w|=r'} \frac{\mathrm{d}w}{w^{-y+1}} \prod_{i=1}^{N} \frac{(a_i w; q)_\infty}{(a_i z; q)_\infty} \prod_{j=1}^{M} \frac{(b_i/z; q)_\infty}{(b_i/w; q)_\infty} \kappa(z, w), \tag{5.10}$$

where $x, y \in \mathbb{Z}$ and

$$\kappa(z,w) = \sqrt{\frac{w}{z}} \frac{(q;q)_\infty^2}{(w/z, qz/w;q)_\infty} \frac{\vartheta_3(\zeta z/w;q)}{\vartheta_3(\zeta;q)}. \tag{5.11}$$

Here ϑ_3 is the theta function. The integration contour's radii satisfy $1 < r/r' < q^{-1}$ and $b_{\max} \le r, r' \le 1/a_{\max}$. Combining this with (5.4), one finds a Fredholm determinant formula for the distribution of the position of the N-th particle in q-PushTASEP. It is

$$\mathbb{E}\left[1/(-tq^{\frac{1}{2}+n-\mu_1};q)_\infty\right] = \det(1-K)_{\ell^2(\mathbb{Z}_{>s})}, \tag{5.12}$$

where K is given by (5.10). Asymptotics is also standard.

6. Application to O'Connell-Yor polymer

By taking appropriate $q \to 1$ limits of both sides in (5.12), we can arrive at the Fredholm determinant formula for the Laplace transform for the O'Connell-Yor polymer, given in (2.4).

6.1. **From q-Push TASEP to O'Connell-Yor polymer.** On the left hand side, we consider $q \to 1$ limit of q-PushTASEP. In fact one can arrive at the O'Connell-Yor polymer in two steps. First one takes a certain $q \to 1$ limit to get the Log-Gamma polymer model, as discussed in [MP17]. This is another finite temperature directed polymer on a lattice with the energy on each site obeys the log-Gamma distribution. Then if one takes an appropriate continuous limit in one direction, one finds the O'Connell-Yor polymer. In the same limit the q-Laplace transform in the left hand side of (5.12) tends to the Laplace transform.

6.2. **Limit of the Fredholm determinant.** We take the corresponding $q \to 1$ limit for the Fredholm determinant on the right hand side of (5.12). The limit to the formula for the Log-Gamma polymer was already studied in [IMSc]. The kernel for this case is very similar to (2.5) with $e^{z^2t/2-w^2t/2}$ replaced by $(\Gamma(B-z)/\Gamma(B-w))^M$. It is easy to check that in the corresponding limit this tends to the ratio of the Gaussian.

7. Conclusion

In this article, we have given a short explanation of our new approach to study KPZ models initiated recently in [IMSa, IMSb, IMSc]. It uses a bijective correspondence between the q-Whittaker measure and the periodic Schur measure, which is associated with a free fermion at finite temperature. Compared to the standard approach using Markov duality and Bethe ansatz, our method has the advantage that Fredholm determinant formulas are obtained by standard machinery of DPPs without going through involved calculations. The kernel has a clear relation to the free fermion and its asymptotics can be studied straightforwardly.

In this article, we mainly focused on a particular directed polymer model, O'Connell-Yor polymer, but we emphasize that our approach is applicable to all solvable KPZ models which are associated with q-Whittaker measures. For example, the asymmetric simple exclusion process (ASEP), which is the best known model in the KPZ class, can be studied by our methods, because the ASEP is known to be related to the Hall-Littlewood measure [BBW18], and the Hall-Littlewood function may be understood as a particular specialization of the q-Whittaker function.

One of the biggest advantages of our approach compared to conventional ones is that one can also study half space models in a parallel fashion. Compared to models in full-space, half-space models have turned out to be difficult to handle with the conventional methods[BBCW18, BBC20]. In particular the proof of the liming distribution for the case of KPZ equation has not been given. In our approach, we can study half-space models by putting a symmetry on our bijection, similarly to the zero temperature case [BR00], and establish the limit theorems for the KPZ equation in [IMSc].

There are still many possible directions to develop this new approach. Studying other statistics such as multi-point joint distributions is one of the most important issues. Understanding similarity and relation to classical discrete integrable systems such as the Box and Ball systems would be also very interesting.

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Appendix A. Determinantal point process and free fermions

A.1. **Determinantal point process.** A determinantal point process (DPP) is a point process for which k point correlation functions are written in the form of determinant,

$$\det(K(x_i, x_j))_{1\le i,j\le k}, \tag{A.1}$$

for all k with the same kernel $K(x,y)$, called the correlation kernel. The distribution of the right most particle is written as a Fredholm determinant,

$$\mathbb{P}[x_1 \le s] = \det(1+K) \tag{A.2}$$

with the same correlation kernel. See for instance [Sos00, ST03, Bor09].

A.2. **Free Fermions.** A free fermion is a quantum many (infinite) particle system for which each one particle state $\phi_n(x)$ ($n \ge 1$, energy ϵ_n) can be either occupied or empty (Pauli principle).

At $T = 0$, for N particles, the ground state filling $n = 1, \dots, N$ is realized. The measure of particle positions is given by

$$\frac{1}{Z}\left(\det(\phi_n(x_m))_{n,m=1}^N\right)^2. \tag{A.3}$$

The position of particles form a DPP, with correlations described by the kernel $K(x,y) = \sum_{n=1}^N \phi_n(x)\phi_n(y)$.

In a system with a positive temperature $T > 0$, the state n is filled with prob $\frac{1}{1+e^{\beta(\epsilon_n-\mu)}}, \beta = \frac{1}{k_B T}$. The position of particles form again a DPP, with its correlation kernel now given by $K(x,y) = \sum_{n=1}^\infty \frac{\phi_n(x)\phi_n(y)}{1+e^{\beta(\epsilon_n-\mu)}}$.

Appendix B. Discrete polymer model at zero temperature and TASEP

In this appendix we explain a discrete analogue of the directed polymer at zero temperature. The connection to TASEP is also mentioned.

B.1. **Discrete directed polymer.** Here we consider a discrete analogue of the relations explained in section 3. Let us now consider a rectangle of size $N \times M$ with grids of unit length. On all lattice sites we put independent random variables and the random variable on site (j,k) is a geometric random variable with parameter $a_j b_k$ where $a_j, b_k > 0, 1 \leq j \leq N, 1 \leq k \leq M$. We consider an up-right path starting from (1,1) and end at (N,M) and regard it as a directed polymer in random environment. The energy of the polymer is a sum of the random variables along the polymer. At zero temperature, we are interested in the maximal energy,

$$G_{N,M} = \max_{\substack{\text{up-right paths from} \\ (1,1)\text{to}(N,M)}} \Big(\sum_{\substack{(i,j) \\ \text{on a path}}} w_{i,j} \Big). \tag{B.1}$$

A sample of a collection of random variables on the rectangle can be regarded as a matrix of size $N \times M$ whose elements are non-negative integers. The Robinson-Schensted-Kunuth (RSK) correspondence [Ful97, Sta99, Sag00] tells us that there is a bijection between such a matrix and a pair (P,Q) of skew Young tabuleaux with the same shape. Besides, in this correspondence, $G_{N,M}$ is equal to λ_1. We are interested in the distribution of $G_{N,M}$, which after the RSK correspondence, corresponds to taking sum with the restriction on the length of λ_1. The sum over semi-standard tableaux with a given shape appearing in this correspondence exactly matches the well-known combinatorial definition of the Schur function,

$$s_\lambda(a) = \sum_{T \in \mathrm{SST}(\lambda)} a^T, \tag{B.2}$$

where $a^T = \prod_i a_i^{\#i \text{ in } T}$ and $\mathrm{SST}(\lambda)$ is the set of semistandard Young tabuleaux with shape λ. We find

$$\mathbb{P}[G_{N,M} \leq u] = \frac{1}{Z} \sum_{\lambda, \lambda_1 \leq u} s_\lambda(a) s_\lambda(b). \tag{B.3}$$

For the Schur function there exists a different expression as a single determinant, known as the Jacobi-Trudi formula, which reads $s_\lambda(a) = \det(h_{\lambda_i - i + j}(a))$, with h_n the complete homogeneous symmetric polynomial. Then the Schur measure is DPP associated with a free fermion at zero temperature. Then using the standard methods of DPP, the probability can be written as a Fredholm determinant and by doing asymptotic analysis one can prove that the limiting law is the Tracy-Widom distribution [Joh00].

By taking a certain limit for the model, one can consider a directed polymer with exponential distributions. One can further take a continuous limit in one direction in which a sequence of random variables becomes a Brownian motion, to get the O'Connell-Yor polymer at zero temperature. From the discrete directed polymer one can take

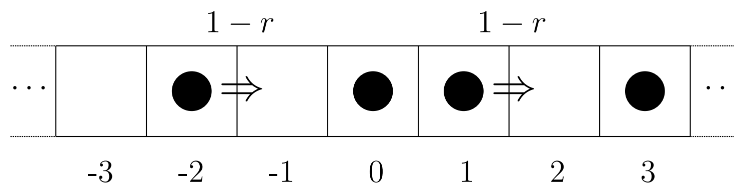

Figure 3. TASEP

another limit, in which one now considers a rectangle of continuous coordinates on which some number of points are Poisson distributed.

B.2. **TASEP.** Here we mention that the discrete direct polymer is intimetely related to totally asymmetric simple exclusion process (TASEP). In a discrete time version of TASEP with parallel update, at each time step, each particle on $\mathbb{Z}$ tries to hop to the right neighboring site with probability $1-r, 0<r<1$, if the target site is empty. See Fig. 3. Let us take the step initial condition in which all non-positive sites are occupied and positive sites are empty at $t=0$, and consider the integrated current $N(t)$ at $(0,1)$ up to time t, namely the number of particles on positive sites at time t.

Let $w_{i,j}$ denote the waiting time for the j-th particle to make the i-th hop since the target site becomes empty. They are independent and identically distributed random variables obeying geometric distribution with parameter r. These $w_{i,j}$'s are nothing but the random environments for the polymer explained above where all a_i's and b_j's. In this correspondence we have $\mathbb{P}[G_{N,N} \le u] = \mathbb{P}[N(t) \ge N]$ [Joh00]. For a more pedagogical account, see for instance [Sas07].

One can consider a few limits. First one can consider continuous time version of TASEP, which is more standard in probability theory and statistical mechanics. In the language of polymer, this corresponds to the case of exponential weight. Corresponding to the piecewise linear Poisson polymer, there is another KPZ model known as the PNG model. This is related to standard Young tableaux.

The relation between TASEP and Schur measure can be also seen in terms of Gelfand-Tsetlin pattern, see [WW09].

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